Infinite Eigenvalue Method for Stability Analyses of Canonical Linear Systems with Periodic Coefficients

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This paper presents a technique that provides both necessary and sufficient conditions for stability of canonical linear differential equations with periodic coefficients and is well-suited for numerical application. Existing methods of stability analyses, not subject to small parameter restrictions, such as the widely used Floquet Numerical Integration Method (FNIM) and the Infinite Determinant Methods provide necessary conditions but are impractical for providing sufficient conditions for this class of systems with multiple degrees of freedom. Furthermore, the Infinite Determinant Methods are restricted to only certain classes of linear periodic systems whereas the FNIM is subject to numerical problems when dealing with canonical systems. The method proposed in this paper, referred to as the Infinite Eigenvalue Method (IEM), is based on Floquet theory and the Infinite Determinant Methods. It is applicable to the general class of multiple degree of freedom linear canonical systems and is not subject to small parameter restrictions. A brief review of Floquet theory and the FNIM is given. The IEM is then presented and applied to a spacecraft dynamics problem of current interest. The FNIM is applied to the same problem and the two methods are compared.

Nomenclature

a	$= \{[k_t(\hat{\nu}^2 + 3/2)]/[4(1+\hat{\nu})]^2\}$
$arg(\lambda_j)$	= argument of the jth characteristic root of a linear periodic system
$[A]$	= infinite coefficient matrix
b	= distance between spacecraft spin axis and nutation damper
c	$= \{[(1-k_t)\hat{\nu}]/[2(1+\hat{\nu})]\}$
c_d	= damping coefficient of nutation damper
$\hat{c}_d$	$= (c_d/m_d\Omega)$
$\hat{c}_{d\nu}$	$= [\hat{c}_d I_{td}/2(1+\hat{\nu})]$
$C_{n,j}$	= complex coefficient multiplying the nth harmonic in the Fourier series representation of the jth almost periodic solution
i	$= \sqrt{-1}$
I_a	= principal mass moment of inertia about the spin axis
I_t	= mass moment of inertia about a principal axis orthogonal to the spin axis
$\hat{I}_{td}$	$= (m_d b^2/I_t)$
k_d	= spring constant of nutation damper
$\hat{k}_d$	$= (k_d/m_d\Omega^2)$
$\hat{k}_{d\nu}$	$= \{[(\hat{k}_d + 1)\hat{I}_{td}]/[4(1+\hat{\nu})^2]\}$
k_j	= number of independent eigenvectors associated with the jth characteristic root λ_j
k_t	$= (I_a - I_t/I_t)$
m	= total mass of spacecraft
m_d	= nutation damper mass
m_j	= multiplicity of the jth characteristic root λ_j
$\hat{m}_d$	$= (m_d/m)$
$\hat{m}_{1d}$	$= (1-\hat{m}_d)\,\hat{I}_{td}$
$[M]$	= multiplicative matrix of a linear periodic system
n	= the number of harmonic terms retained in the Fourier series' representation of the almost periodic solution (also referred to as the truncation level)
O	= mass center of undeformed configuration
$OA_1A_2A_3$	= orbiting axes which rotate with a constant absolute angular rotation rate ν about the A_2 axis normal to the orbit plane
$Ox_1x_2x_3$	= body fixed axes coincident with the principal axes
$[P(t)]$	= periodic coefficient matrix of a linear periodic system
q	$= [3k_t/16(1+\hat{\nu})^2]$
r	= magnitude of the spacecraft position vector
s	$= (arg\{\lambda_j\}/T\omega)$
t	= time
T	= period of the coefficients in a linear periodic system
w	$= \{(\hat{I}_{td}\hat{\nu}^2)/[4(+\hat{\nu})^2]\}$
$\{x\}$	= state vector for a linear system
$\{y\}$	= infinite Fourier series coefficient vector
Z	$= e^{i\omega t}$
$\alpha_1,\alpha_2,\alpha_3$	= Euler angles, depicted in Fig. 2
θ	$= 2(1+\hat{\nu})\Omega t$; independent variable
λ_j	= the jth characteristic root of a linear periodic system
ν	= absolute spin rate of the $OA_1A_2A_3$ axes
$\hat{\nu}$	$= (\nu/\Omega)$
ξ	= excursions of the nutation damper mass
η	$= (\xi/b)$
$\phi(t)$	= an arbitrary T periodic function; $\phi(t+T) = \phi(t)$
$[\Phi(t)]$	= fundamental solution matrix for a linear periodic system
ω	$= (2\pi/T)$
$\omega_1,\omega_2,\omega_3$	= component absolute angular velocities of the $Ox_1x_2x_3$ axes
Ω	= angular rate of the circular orbit
$(\dot{\text{u}})$	$= (d/dt)$
$(\ )'$	$= (d/d\theta)$

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Introduction

DYNAMIC modeling of many engineering systems leads to a system of linear ordinary differential equations with periodic coefficients (linear periodic systems). Some typical examples are

1) rotating systems, such as wind turbines and helicopter rotor blades[1]
2) elastic structures under parametric excitation[2-8]
3) spin-stabilized and dual spin spacecraft[9-19].

In the dynamic analysis of an engineering system, two basic problems that require investigation are

1) the stability of the system
2) the response of the system under various types of excitation.

This paper addresses the first problem and focuses specifically on those systems modeled by linear periodic systems.

Stability of linear periodic systems has received considerable attention in the past. Although a complete review of the literature is beyond the scope of this study, the references already cited provide a good cross section of available methods of stability analysis. These can be divided into the following categories:

1) Perturbation Methods[4,5,20]
2) Infinite Determinant Methods[2,7,8,14,15,21]
3) Floquet Numerical Integration Method[3,6,9,11,12,13,17,18]

Note that the methods that transform the periodic equations to constant coefficient equations[1,19] are not included in this study, as the same stability analysis techniques used for constant coefficient systems are implemented after the transformation is made. Furthermore, these methods are not applicable to the majority of the periodic systems.

Perturbation methods are based on the assumption that the periodic coefficient terms are small. Their main advantage is that they provide approximate closed-form expressions for the transition curves that define the stability boundaries in the parameter space. These methods typically require a great deal of algebra, however, and are limited by the small parameter restriction. A review of perturbation methods and their applications is given by Nayfeh and Mook.[20] Infinite Determinant Methods are generally not subject to the small parameter restriction; however, they too are approximate, as the truncation of a converging determinant is required. Bolotin[2] used Hill's method of infinite determinants extensively for scalar equations. He provided an extension for multiple degree of freedom systems with certain restrictions placed on the damping matrix, but found that the equations became too cumbersome and, therefore, impractical. Lindh and Likins[14] later extended Bolotin's method for completely damped multiple degree of freedom systems, using numerical techniques to solve the determinant. A completely damped system has the characteristic that the motion of each state variable in the state space is influenced by a damping force. They used their method to investigate the stability of a dissipative dual-spin satellite previously investigated by Mingori[18] using the Floquet Numerical Integration Method (FNIM). Their studies found that, although the method accurately determined the stability boundaries, it was more cumbersome to formulate than the Floquet method used by Mingori and required more computation time. Takahashi[8] recently applied Bolotin's equations for multiple degree of freedom systems by converting the determinant to an eigenvalue problem and solving numerically.

The FNIM is based on the Floquet theory. It is the most widely used technique, as it is applicable to the entire class of linear periodic systems. Its main deficiency is its high computational demands. To reduce these demands, Hsu[6] and Friedmann et al.[3] have developed numerically efficient schemes that significantly reduce the number of required numerical integrations. The FNIM provides both necessary and sufficient conditions for stability of completely damped multiple degree of freedom systems.

The focus of this study is the stability of canonical linear periodic systems that can be written in the following form:

$$[M]\{\ddot{x}\} + [G]\{\dot{x}\} + [K]\{x\} = \{0\} \tag{1}$$

where $[M]^T = [M]$, $[G]^T = -[G]$ and $[K]^T = [K]$. The coefficient matrices $[M]$, $[G]$, and $[K]$ are associated with inertial, gyroscopic, and stiffness forces, respectively. In this context, the term *canonical system* refers to those systems that are not subjected to damping forces (i.e., conservative systems).

For the class of systems depicted by Eq. (1), the FNIM suffers from two deficiencies. The first is that the FNIM readily provides necessary conditions for stability but is not well suited for providing sufficient conditions. Under certain circumstances, which will be discussed in a later section, a cumbersome investigation of the governing equations for the FNIM is required to yield the sufficient conditions. The second deficiency, perhaps of even more importance, is the numerical problem encountered when dealing with this class of systems, as the stability criterion involves an equality condition that is impossible to satisfy numerically. To overcome this problem in practice, the stability criterion must be relaxed to form an inequality condition that is then suitable for numerical application. The difficulty with this approach is choosing th appropriate degree of relaxation so that erroneous conclusions concerning stability will not result.

Linear periodic equations that model many engineering systems, particularly those in spacecraft dynamics, are obtained through a process of linearization of a set of nonlinear equations. In this case, the stability properties of the linearized or variational equations are said to be the *infinitesimal* stability properties of the true nonlinear system. The nonlinear inferences, appropriate to periodic systems, of infinitesimal stability properties are as follows:

1) If the null solution of the linearized system is asymptotically stable, then so too is the null solution of the true nonlinear system.

2) If the null solution of the linearized system is unstable, then so too is the null solution of the true nonlinear system.

For proofs of the above two statements, refer to Coddington and Levinson.[24] Since this study deals specifically with canonical linear periodic systems, asymptotic stability is not possible and, hence, the first of the above statements does not apply. Therefore, stability of the nonlinear system cannot be inferred from the stability of the linearized system. Consequently, for those engineering systems modeled by nonlinear equations and approximated by canonical linear periodic systems, the methods of stability analysis discussed in this paper only identify regions of instability in the parameter space. Although, this is a relatively weak statement of stability, these methods are generally used as a first step to the more complex analysis of the nonlinear equations.

The purpose of this paper is to present a method of stability analysis for the general class of canonical linear periodic systems that is free from small parameter approximations and circumvents the deficiencies of the FNIM. That is, it simultaneously provides both the necessary and sufficient conditions for stability and is well suited for numerical application. The method, referred to as the Infinite Eigenvalue Method (IEM), is based on Floquet theory and the Infinite Determinant Methods. In the sections that follow, a brief review of both the Floquet theory and the FNIM is given. The IEM is then presented and applied to a spacecraft dynamics problem of current interest. The FNIM is applied to the same problem and the results of the two methods are compared.

Floquet Theory and Floquet Numerical Integration Method

Linear periodic systems can be expressed in the following general form:

$$\{\dot{x}\} = [P(t)]\{x\} \tag{2}$$

where $[P(t)]$ is an $N \times N$ periodic coefficient matrix, with $[P(t+T)] = [P(t)]$. Let the matrix $[\Phi(t)]$ be the fundamental solution matrix where each column is an independent solution to the periodic system [Eq. (2)]. Therefore, $[\Phi(t)]$ must satisfy

$$[\dot{\Phi}(t)] = [P(t)][\Phi(t)] \tag{3}$$

Since the coefficient matrix $[P(t)]$ is periodic, $[\Phi(t+T)]$ must also be a fundamental solution matrix. Therefore, since all solutions of linear systems must be linear combinations of the fundamental solutions, the following equation must be satisfied:

$$[\Phi(t+T)] = [\Phi(t)][M] \tag{4}$$

Matrix $[M]$ is constant, nonsingular, of dimension $N \times N$, and is referred to as the multiplicative matrix. Equation (4) is the basis of Floquet theory.

It can be shown that all solutions to Eq. (2) will have one of two general forms, referred to as the type 1 and type 2 solutions.[22] The type 1 solutions are also referred to as normal solutions,[21] or Floquet solutions.[1] The existence of either type 1 or type 2 solutions depends upon the characteristic roots λ_j of $[M]$, their multiplicity m_j, and the number of independent characteristic vectors k_j of the repeated characteristic roots. The conditions governing the existence of either type 1 or type 2 solutions are summarized below in terms of the solutions corresponding to λ_j.

$m_j = 1$	a type 1 solution will exist
$m_j > 1$ and $m_j = k_j$	m_j type 1 solutions will exist
$m_j > 1$ and $m_j > k_j$	both type 1 and type 2 solutions will exist

The form of the type 1 and type 2 solutions are as follows:[22]

Type 1 Solutions

$$x_{j+p}(t) = e^{\frac{\ln(|\lambda_j|)}{T}t} e^{i\frac{arg(\lambda_j)}{T}t} \phi_{j+p}(t) \qquad 1 \le p \le k_j \tag{5}$$

Type 2 Solutions

$$x_{k_j+p}(t) = e^{\frac{\ln(|\lambda_j|)}{T}t} e^{i\frac{arg(\lambda_j)}{T}t} [\phi_{k_j+p}(t) + \frac{t}{T\lambda_j}\phi_{k_j+p-1}(t) + \frac{t(t-1)}{2T^2\lambda_j^2}\phi_{k_j^2+p-2}(t) + \cdots$$
$$\frac{t(t-T)\ldots[t+(1-p)T]}{p!T^p\lambda_j^p}\phi_{k_j}(t) \qquad 1 \le p \le m_j - k_j \tag{6}$$

where $|\lambda_j|$ is the modulus and $arg(\lambda_j)$ is the argument of the characteristic root λ_j, and $\phi_j(t)$ is a periodic function, with $\phi_j(t+T) = \phi_j(t)$. The above forms of solution lead to the following criteria for stability.[22]

1) If the modulus $|\lambda_j|$ of at least one solution exceeds unity, then that solution is unbounded and the linear periodic system (2) is unstable.

2) If the modulus $|\lambda_j|$ of all characteristic roots is less than unity, then the linear periodic system (2) is asymptotically stable.

3) If the modulus $|\lambda_j|$ of all characteristic roots is less than or equal to unity, then the linear periodic system (2) is stable but not asymptotically stable, if and only if type 1 solutions correspond to all roots with modulus equal to unity.

4) If the modulus $|\lambda_j|$ of all characteristic roots is less than or equal to unity, then the linear periodic system (2) is unstable if there exists at least one type 2 solution corresponding to a root with modulus equal to unity.

It follows that the stability of a linear periodic system can be investigated by determining the characteristic roots of the multiplicative matrix $[M]$, provided all roots with a modulus equal to unity are distinct. If repeated roots with a unit modulus exist, the characteristic vectors corresponding to the repeated roots must also be determined to establish if the linear periodic system is stable.

The FNIM is a direct application of Floquet theory; it constructs the multiplicative matrix and finds its characteristic roots. To construct $[M]$, consider a solution to the linear periodic system after the first period:

$$\{x(T)\} = [M]\{x(0)\} \tag{7}$$

By choosing an initial condition such that the first element of the initial condition vector is unity while all other elements are zero, the solution after one period is equal to the first column of the multiplicative matrix. In general, if the initial condition vector has the jth element set to unity and all other elements set to zero, the solution after one period is equal to the jth column of the multiplicative matrix. These types of initial conditions are referred to as the Floquet initial conditions.[1] Therefore, if a numerical integration is performed to yield a solution after the first period for each of the independent Floquet initial conditions, the multiplicative matrix can be constructed. Once the multiplicative matrix is constructed, a numerical eigenvalue routine can be used to find its characteristic roots. The stability criteria listed above are applied to each characteristic root to determine if the linear periodic system is stable. To avoid N numerical integrations of the linear periodic system, Hsu[6] and Friedmann et al.[3] developed alternate methods that approximate the multiplicative matrix after only one numerical integration. These methods are well suited for large order systems where the increase in numerical efficiency is apparent.

Canonical linear periodic systems have the unique characteristic that if λ_j is a characteristic root, then $(1/\lambda_j)$ must also be a characteristic root.[23] Therefore, a canonical system can be either stable or unstable, but never asymptotically stable. Here, a necessary condition for the stability of canonical systems is that all characteristic roots have a modulus equal to unity. It is this information that is used when investigating the stability using the FNIM. However, according to the stability criteria outlined above, an additional requirement for stability is that only type 1 solutions exist. This stability requirement is satisfied if all the characteristic roots of $[M]$ are distinct. In the event repeated roots exist, the characteristic vectors of the repeated roots must be examined.

The FNIM is, however, subject to numerical difficulties when dealing with this class of systems. Since numerical methods are used to evaluate the characteristic roots, it is not possible to determine if the roots have modulus of exactly unity (an equality condition). To deal with this problem in practice, the stability condition requiring all characteristic roots to lie on the unit circle in the complex plane is relaxed by allowing them to lie in a small band around the unit circle (an inequality condition). The difficulty with this approach is determining the appropriate size of the band. Similarly, it is difficult to numerically assess if there exist repeated roots, due to the inherent numerical approximation.

Infinite Eigenvalue Method

To begin the formulation of the IEM, substitute $|\lambda_j| = 1$, the necessary condition for stability of canonical linear periodic systems, into the general form of the type 1 solution, [Eq. (5)]

$$x_j(t) = e^{i\frac{arg(\lambda_j)}{T}t}\phi_j(t) \qquad 1 \le j \le N \tag{8}$$

where $i = \sqrt{-1}$, and $\phi_j(t+T) = \phi_j(t)$. This form of solution is termed the almost periodic solution.[14] By ensuring that all solutions of the linear periodic system have the form of Eq. (8), both the necessary and sufficient conditions for stability are satisfied. This is the basis of the IEM, to determine those combinations of system parameters that permit N solutions of the form given in Eq. (8), where N is the order of the periodic coefficient matrix $[P(t)]$.

To formulate the governing equations for the IEM, the almost periodic solution is first expressed in terms of a complex Fourier series

$$x_j(t) = e^{i\frac{arg(\lambda_j)}{T}t} \sum_{n=-\infty}^{\infty} C_{n,j} e^{in\omega t} \tag{9}$$

where

$$\omega = (2\pi/T) \tag{10}$$

$$C_{n,j} = a_{n,j} + ib_{n,j} \tag{11}$$

By defining the following parameters

$$s = \frac{arg(\lambda_j)}{T\omega} \tag{12}$$

$$Z = e^{i\omega t} \tag{13}$$

and substituting into Eq. (9), the form of the almost periodic solution becomes

$$x_j(t) = \sum_{n=-\infty}^{\infty} C_{n,j} Z^{(n+s)} \tag{14}$$

This form of the almost periodic solution is more amenable to the formulation of the governing equations for the IEM.

Equation (14) is substituted into the linear periodic system and the harmonics are balanced to form a set of linear algebraic equations for the Fourier coefficients $C_{n,j}$. This set of linear algebraic equations will have the form

$$[A]\{y\} = \{0\} \tag{15}$$

Matrix $[A]$ is an infinite coefficient matrix, and $\{y\}$ is the vector of the Fourier coefficients. Since only the nontrivial solutions are of interest, then the vanishing of the determinant of matrix $[A]$ is a condition for the existence of at least one almost periodic solution. This determinant is referred to as an infinite determinant. The problem of searching over the parameter values to find the combinations that will cause the determinant to vanish is the essence of the infinite determinant methods of stability analysis. Since the infinite determinants belong to a class of converging determinants,[2] they are typically truncated after several terms in the Fourier series to yield approximate results.

The IEM converts the infinite determinant to the following eigenvalue problem by expressing Eq. (15) explicitly in terms of the scalar s.

$$det[(B) - s(1)] = \{0\} \tag{16}$$

Generally, the infinite matrix $[B]$ is truncated, or equivalently, the Fourier series representations of the almost periodic solutions are truncated after the nth harmonic term. The order n of the Fourier series required for sufficient accuracy depends on the rate of convergence of the Fourier series expansions. One advantage of using the form of the almost periodic solution given by Eq. (14) is that matrix $[B]$ can be formulated before deciding on a truncation level. Hence, the truncation level n becomes a user-specified variable in the computer algorithm. A suitable value of n can be determined by examining the normalized eigenvectors of the above eigenvalue problem [Eq. (16)].

The eigenvalue problem must initially be solved for one point in the parameter space with a relatively high truncation level, and the normalized eigenvector must be computed. By examining the relative magnitudes of the components of the eigenvectors corresponding to the Fourier coefficients, a more suitable lower truncation level can be established. This truncation level can then be used to test the desired points in the parameter space for stability, providing the parameters do not significantly affect the convergence characteristic of the Fourier series expansions.

By converting the infinite determinant to an eigenvalue problem, it is possible to determine the number of almost periodic solutions present. Recall that the necessary and sufficient condition for stability is that N almost periodic solutions must exist. The real eigenvalue s of the infinite eigenvalue problem, Eq. (16), is related to the argument of the characteristic root λ_j by

$$s_j = \frac{arg(\lambda_j)}{2\pi} \tag{17}$$

From the above equation, it is evident that the real interval $-\frac{1}{2} \le s \le \frac{1}{2}$ corresponds to the interval $-\pi \le arg(\lambda) \le \pi$, which uniquely defines the set of characteristic roots associated with almost periodic solutions that lie on the unit circle. Consequently, the existence of N real eigenvalues s in the interval $-\frac{1}{2} \le s \le \frac{1}{2}$ implies that there must exist N almost periodic solutions to the linear periodic system.

Due to this relationship between s and λ_j, described by Eq. (17), a property of the eigenvalues s of the eigenvalue problem Eq. (16) is that an arbitrary real eigenvalue s can be defined as

$$s = s_j + k \tag{18}$$

where s_j is a real eigenvalue in the interval $-\frac{1}{2} \le s \le \frac{1}{2}$, and k is an arbitrary integer. Hence, the existence of N real eigenvalues s in any unit interval along the real axis implies there must exist N almost periodic solutions to the periodic system. However, using the interval $-\frac{1}{2} \le s \le \frac{1}{2}$, (i.e., $k = 0$) was found to minimize the truncation error. Another property of the real eigenvalues s of Eq. (16), due to Eq. (17), is that they are symmetric about the origin, since all characteristic roots must have a complex conjugate. Hence, only the interval $0 \le s \le \frac{1}{2}$ needs to be examined.

In summary, the IEM involves constructing the infinite eigenvalue [Eq. (16)], truncating at the appropriate level, and determining its eigenvalues. The linear periodic system is stable if and only if there exist $N/2$ real eigenvalues s in the interval $0 \le s \le \frac{1}{2}$.

Application: Spin-Stabilized Satellite with Nutation Damping in a Circular Orbit

In light of continuing interest in small spin-stabilized spacecraft at Bristol Aerospace, the stability of the linear periodic equations modeling the infinitesimal dynamical behavior of the spacecraft proposed for the POLARIS (POLAR Ionospheric Studies) strawman mission is investigated. A schematic of the spacecraft is given in Fig. 1. One of the mission requirements is a circular low-Earth polar orbit with the spin axis of the spacecraft nominally aligned with the orbit normal. Assuming that the gravity gradient is the only source of external disturbance, there will be no regression of the line of nodes when the spacecraft is in a polar orbit. Hence, this orientation corresponds to an equilibrium motion. More specifically, this equilibrium is termed the Thompson equilibrium, following a classical paper by Thompson[19] that investigated the stability of a rigid axisymmetric spinning spacecraft in a circular orbit with the spin axis aligned with the orbit normal. The linearized equations of motion for Thompson's problem have constant coefficients. The POLARIS spacecraft differs from Thompson's system in that it includes a nutation damper to damp the nutation about the angular momentum vector. The nutation damper complicates the linearized motion equations by introducing periodic coefficients. Consequently, Thompson's investigation is not applicable. By setting the damping coeffi-

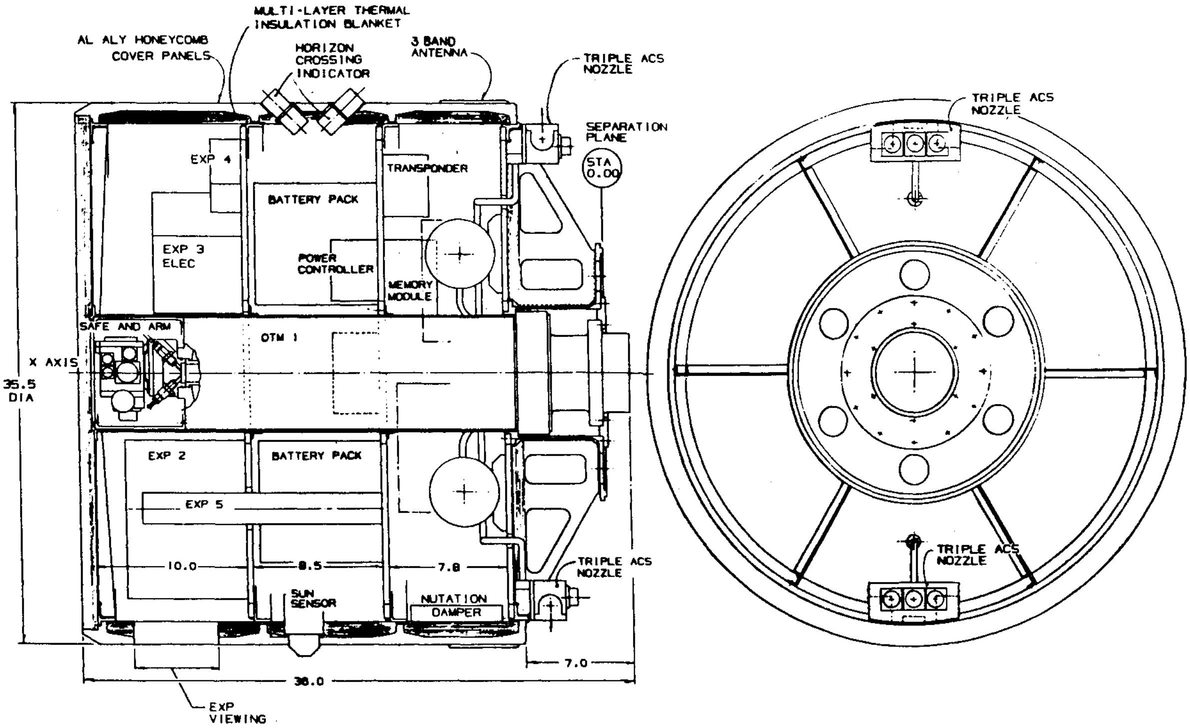

Fig. 1 Schematic drawing of the POLARIS spacecraft.

cient for the nutation damper to zero, the linear periodic equations of motion can be put into canonical form. Hence, these equations provide a suitable application for the IEM. For verification of the method, both the IEM and the FNIM are applied to construct stability diagrams in the parameter space.

The mathematical model for the POLARIS spacecraft is shown in Fig. 2. The axial nutation damper is modeled as a spring-mass-dashpot system, with the point mass located on a principal axis when in the equilibrium position. In the undeformed configuration, the satellite is assumed to be inertially axisymmetric about the x_2 axis. The satellite body fixed axes $(Ox_1x_2x_3)$, defined to be coincident with the principal axes, are referenced to orbiting axes $(OA_1A_2A_3)$ by a set of Euler angles $(\alpha_1\alpha_2\alpha_3)$ generated by a right-hand rotation scheme. Axes $(OA_1A_2A_3)$ rotate with a constant absolute rotation rate ν about the A_2 axis normal to the orbit plane. The angular velocity of the circular orbit is designated by Ω.

The equilibrium motion of interest corresponds to a pure spin ν about the A_2 axis, that is, axes $(Ox_1x_2x_3)$ are coincident with axes $(OA_1A_2A_3)$. This implies

$$\omega_1 = \omega_3 = \xi = 0 \qquad \omega_2 = \nu \tag{19}$$

where ω_1, ω_2, ω_3 are components of the spacecraft absolute angular velocity. Assuming infinitesimally small angular deviations about the equilibrium motion, the equations relating the Euler angles to the absolute angular velocities are

$$\omega_1 = \dot{\alpha}_1 + \nu\alpha_3 \qquad \omega_2 = \dot{\alpha}_2 + \nu \qquad \omega_3 = \dot{\alpha}_3 + \nu\alpha_1 \tag{20}$$

The motion equations that describe the dynamic behavior of this four degree of system $(\alpha_1, \alpha_2, \alpha_3, \xi)$ are derived and linearized about the solution in Eq. (19). By inspecting the equations, it is readily observed that the system is unstable with respect to perturbations of the spin rate $\dot{\alpha}_2$. This is inherent with spin-stabilized spacecraft, as there is generally no restoring torque about the spin axis. However, the equation for the spin perturbation is uncoupled from the remaining motion equations and can, therefore, be dropped, allowing an

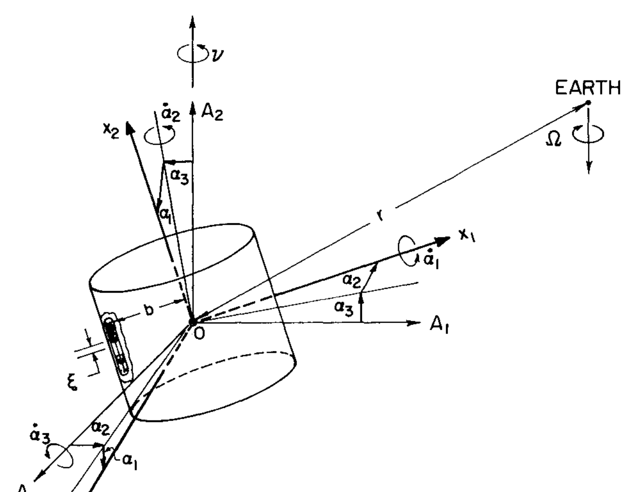

Fig. 2 Mathematical model of a spacecraft with a nutation damper in a circular orbit.

infinitesimal stability investigation of the attitude motions (α_1, α_3) and the excursions of the damper mass (ξ). The motion equations are given below in nondimensional form.

$$\alpha_1'' + c\alpha_3' + (a + 2q\cos\theta)\,\alpha_1 + 2q\sin\theta\alpha_3 = 0 \tag{21a}$$

$$\alpha_3'' - c\alpha_1' + (a - 2q\cos\theta)\,\alpha_3 + 2q\sin\theta\alpha_1 + \tilde{I}_{td}\eta'' + w\eta = 0 \tag{21b}$$

$$\hat{m}_{1d}\eta'' + \hat{c}_{d\nu}\eta' + \hat{k}_{d\nu}\eta + \tilde{I}_{td}\alpha_3'' + w\alpha_3 = 0 \tag{21c}$$

where $\theta = 2(1 + \hat{\nu})\,\Omega t$, $\eta = (\xi/b)$ and the primes denote differentiation with respect to θ. The periodic terms in the above equations arise due to the gravitational torque expressions. By setting the damping coefficient c_d to zero, the linear periodic equations become canonical and, thus, can be investigated for stability using the IEM.

To apply the IEM to Eqs. (21), assume an almost periodic solution, Eq. (14), for the solutions of the canonical system.

$$\alpha_1(\theta) = \sum_{n=-\infty}^{\infty} C_{n,1} Z^{(n+s)} \tag{22a}$$

$$\alpha_3(\theta) = \sum_{n=-\infty}^{\infty} C_{n,2} Z^{(n+s)} \tag{22b}$$

$$\eta(\theta) = \sum_{n=-\infty}^{\infty} C_{n,3} Z^{(n+s)} \tag{22c}$$

Substituting Eq. (22) into (21), applying the relations

$$\frac{dZ}{dt} = i\omega e^{i\omega t} = i\omega Z \tag{23a}$$

$$\frac{d^2Z}{dt^2} = -\omega^2 e^{i\omega t} = -\omega^2 Z \tag{23b}$$

and using Euler's equations

$$2\cos\omega t = e^{i\omega t} + e^{-i\omega t} = Z + Z^{-1} \tag{24a}$$

$$2\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{i} = i(-Z + Z^{-1}) \tag{24b}$$

leads to the following

$$\sum_{n=-\infty}^{\infty} \{[-(n+s)^2 + a]C_{n,1} + ic(n+s)C_{n,2} + qC_{n-1,1} + qC_{n+1,1} - iqC_{n-1,2} + iqC_{n+1,2}\} Z^{(n+s)} = 0 \tag{25a}$$

$$\sum_{n=-\infty}^{\infty} \{[-(n+s)^2 + a]C_{n,2} - ic(n+s)C_{n,1} - qC_{n-1,2} - qC_{n+1,2} + iqC_{n-1,1} - iqC_{n+1,1} + [-\hat{I}_{td}(n+s)^2 + w]C_{n,3}\} Z^{(n+s)} = 0 \tag{25b}$$

$$\sum_{n=-\infty}^{\infty} \{[-\hat{m}_{1d}(n+s)^2 + \hat{k}_{dv}]C_{n,3} + [-\hat{I}_{td}(n+s)^2 + w]C_{n,2}\} Z^{(n+s)} = 0 \tag{25c}$$

Without loss of generality, we can set

$$C_{n,2} = iC_{n,2}$$

$$C_{n,3} = iC_{n,3}$$

and multiply Eqs. (25b) and (25c) by i and Eq. (25a) by -1, to arrive at the following:

$$[(n+s)^2 - a]C_{n,1} + c(n+s)C_{n,2} - qC_{n-1,1} - qC_{n+1,1} - qC_{n-1,2} + qC_{n+1,2} = 0 \tag{26a}$$

$$[(n+s)^2 - a]C_{n,2} + c(n+s)C_{n,1} + qC_{n-1,2} + qC_{n+1,2} - qC_{n-1,1} + qC_{n+1,1} + [\hat{I}_{td}(n+s)^2 - w]C_{n,3} = 0 \tag{26b}$$

$$[\hat{m}_{1d}(n+s)^2 - \hat{k}_{dv}]C_{n,3} + [\hat{I}_{td}(n+s)^2 - w]C_{n,2} = 0 \tag{26c}$$

This simplifies the equations, since i no longer appears explicitly. Expressing Eqs. (26) in matrix form, the infinite matrix equation is constructed

$$[A]\{y\} = \{0\} \tag{27}$$

where

$$[A] = \begin{bmatrix} \infty & & & & & \\ & \ddots & & & & \\ & & [Q]^T\ [R_{n-1}]\ [Q] & & & \\ & & & [Q]^T\ [R_n]\ [Q] & & \\ & & & & [Q]^T\ [R_{n+1}]\ [Q] & \\ & & & & & \ddots \\ & & & & & -\infty \end{bmatrix} \tag{28a}$$

$$[Q] = \begin{bmatrix} -q & q & 0 \\ -q & q & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28b}$$

$$[R_n] = \begin{bmatrix} (n+s)^2 - a & c(n+s) & 0 \\ c(n+s) & (n+s)^2 - a & \hat{I}_{td}(n+s)^2 - w \\ 0 & \hat{I}_{td}(n+s)^2 - w & \hat{m}_{1d}(n+s)^2 - \hat{k}_{dv} \end{bmatrix} \tag{28c}$$

Expressing Eq. (27) explicitly in terms of s gives

$$([A_a]s^2 + [A_b]s + [A_c])\{y\} = \{0\} \tag{29}$$

By defining a vector $\{u\} = s\{y\}$, the quadratic eigenvalue problem can be reduced to standard form

$$([B] - s[1])\{q\} = \{0\} \tag{30}$$

where

$$\{q\} = \begin{Bmatrix} \{u\} \\ \{y\} \end{Bmatrix} \tag{31}$$

$$[B] = \begin{bmatrix} -[A_1] & -[A_2] \\ [1] & 0 \end{bmatrix} \tag{32}$$

$$[A_1] = [A_a]^{-1}[A_b] \tag{33}$$

$$[A_2] = [A_a]^{-1}[A_c] \tag{34}$$

Tyc[22] has shown that the matrix $[A_a]$ is always invertible for canonical linear mechanical systems that have the form

$$[M]\{\ddot{x}\} + [G]\{\dot{x}\} + [K]\{x\} = \{0\} \tag{35}$$

Matrices $[A_1]$ and $[A_2]$ of Eqs. (33) and (34), are given below:

$$[A_1] = \begin{bmatrix} \infty & & & & & \\ & \ddots & & & & \\ & & [A_{1,(n-1)}] & & & \\ & & & [A_{1,n}] & & \\ & & & & [A_{1,(n+1)}] & \\ & & & & & \ddots \\ & & & & & -\infty \end{bmatrix} \tag{36a}$$

where

$$[A_{1,n}] = \begin{bmatrix} 2n & c & 0 \\ \dfrac{c\,\hat{m}_{1d}}{\hat{m}_{1d} - \hat{I}_{td}^2} & 2n & 0 \\ 0 & 0 & 2n \end{bmatrix} \tag{36b}$$

and

$$[A_2] = \begin{bmatrix} \infty & & & & & \\ & \ddots & & & & \\ & & [Q_1][A_{2,(n-1)}][Q_2] & & & \\ & & & [Q_1]\ [A_{2,n}]\ [Q_2] & & \\ & & & & [Q_1]\ [A_{2,(n+1)}][Q_2] & \\ & & & & & \ddots \\ & & & & & & -\infty \end{bmatrix} \tag{37a}$$

where

$$[Q_1] = \begin{bmatrix} -q & -q & 0 \\ \dfrac{\hat{m}_{1d} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & \dfrac{\hat{m}_{1d} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & 0 \\ \dfrac{-\hat{I}_{td} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & \dfrac{-\hat{I}_{td} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & 0 \end{bmatrix} \tag{37b}$$

$$[A_{2,n}] = \begin{bmatrix} n^2 - a & cn & 0 \\ \dfrac{\hat{m}_{1d} cn}{\hat{m}_{1d} - \hat{I}_{td}^2} & n^2 - \dfrac{\hat{m}_{1d} a - \hat{I}_{td} w}{\hat{m}_{1d} - \hat{I}_{td}^2} & \dfrac{\hat{I}_{td} \hat{k}_{dv} - \hat{m}_{1d} w}{\hat{m}_{1d} - \hat{I}_{td}^2} \\ \dfrac{-\hat{I}_{td} cn}{\hat{m}_{1d} - \hat{I}_{td}^2} & \dfrac{\hat{I}_{td} a - w}{\hat{m}_{1d} - \hat{I}_{td}^2} & n^2 - \dfrac{\hat{k}_{dv} - \hat{I}_{td} w}{\hat{m}_{1d} - \hat{I}_{td}^2} \end{bmatrix} \tag{37c}$$

$$[Q_2] = \begin{bmatrix} -q & q & 0 \\ \dfrac{-\hat{m}_{1d} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & \dfrac{\hat{m}_{1d} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & 0 \\ \dfrac{\hat{I}_{td} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & \dfrac{-\hat{I}_{td} q}{\hat{m}_{1d} - \hat{I}_{td}^2} & 0 \end{bmatrix} \tag{37d}$$

The standard eigenvalue problem in Eq. (30) is formulated by substituting equations (36) and (37) into (32). Standard library eigenvalue routines can now be used to determine the eigenvalues. Upon inspection of the eigenvectors of Eq. (15), matrices $[A_1]$ and $[A_2]$, defined by Eqs. (36) and (37), are truncated after the third harmonic in the Fourier series representations of the almost periodic solutions (truncation level $n = 3$).

The infinite matrix $[B]$, therefore, becomes a finite matrix of order 42. The IMSL routine EIGRF is used to numerically solve for the eigenvalues. Recall that the necessary and sufficient condition for stability is that there must exist three real eigenvalues between 0 and (1/2).

Assuming a 300 km circular orbit and the following spacecraft parameters

$m = 400$ lbm $I_t = 26$ slug ft^2 $m_d = 4$ lbm

$k_d = 1.66 \times 10^{-7}$(lbf/ft) $b = 1.45$ ft $c_d = 0$

leads to the following nondimensionalized parameters:

$\hat{m}_d = 0.01$

$\hat{I}_{td} = 0.01$

$\hat{k}_d = 1.0$

$\hat{c}_d = 0.0$

The extremely small value for the spring stiffness of the nutation damper (k_d) is realistic for only very low spin rates. Generally, nutation dampers of this type are dynamically tuned to maximize the energy dissipated per cycle. This requires adjusting the natural frequency of the damper to the expected excitation frequency, which is a function of the spacecraft principal moments of inertia and the spin rate. If, for example, $k_t = 0.2$, then the spin rate required to equate the natural and excitation frequencies of the nutation damper is approximately 0.055 rpm.

Assuming the above nondimensional parameters are fixed, the $k_t - \hat{\nu}$ parameter space is then investigated for stability.

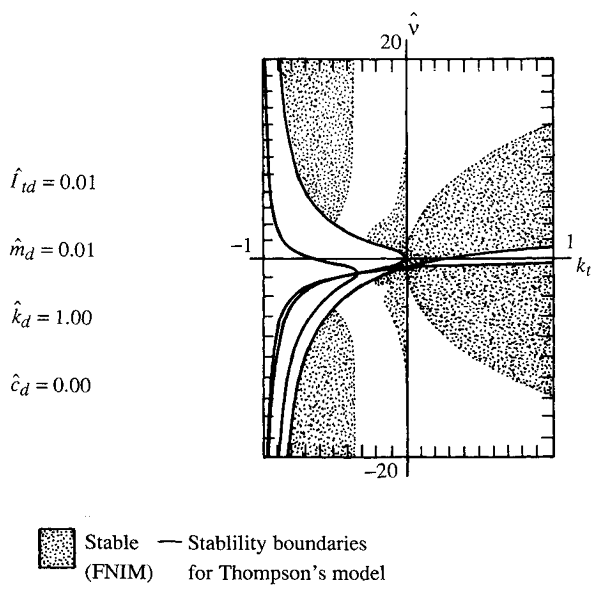

Fig. 3 Stability diagram for a spin stabilized spacecraft with a nutation damper in a circular orbit using the Infinite Eigenvalue Method.

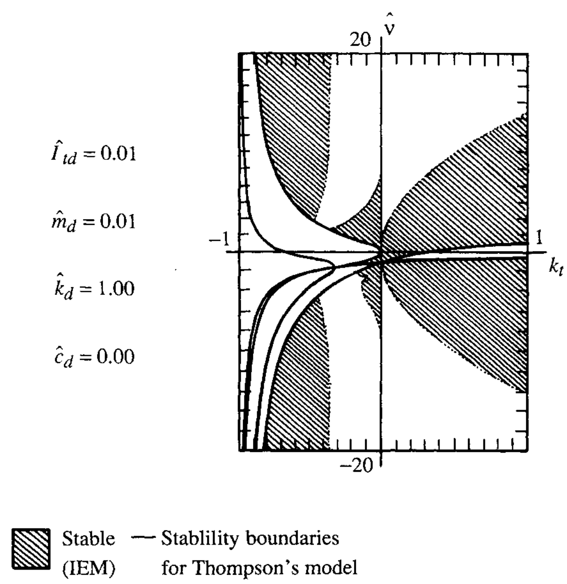

Fig. 4 Stability diagram for a spin stabilized spacecraft with a nutation damper in a circular orbit using the Floquet Numerical Integration Method.

The region of interest in this parameter plane is defined by $-1 \leq k_t \leq 1$, which accounts for all possible configurations of a symmetric real body, and $-20 \leq \bar{\nu} \leq 20$, which corresponds to spin rates between -0.22 and 0.22 rpm for a 300 km circular orbit. The resulting stability diagram is given in Fig. 3. Also included in the stability diagram are the stability boundaries for a spacecraft without a nutation damper, as investigated by Thompson. The stability diagram shows that the addition of the nutation damper has a significant destabilizing effect on the spacecraft. Furthermore, the nutation damper does not stabilize regions originally unstable in Thompson's analysis.

As a verification of the stability diagram given in Fig. 3, the FNIM is used to analyze the same system. Recall that the criterion for stability is that all the characteristic roots of the multiplicative matrix of the periodic system must lie on the unit circle. To compensate for numerical error, a band width of ± 0.003 is introduced around the unit circle. The stability criterion then becomes: all characteristic roots must lie inside this narrow band. Figure 4 presents the stability results and shows that the FNIM reproduces the results of the IEM remarkably well. The fact that the FNIM determined all the regions of stability indicates that the characteristic roots associated with each point investigated in the parameter plane, are either all distinct or include repeated roots, with the number of characteristic vectors equal to the multiplicity of all roots.

Conclusions

The IEM has been presented in this paper. It is applicable to the general class of multiple degree of freedom canonical linear periodic systems. The main conclusions obtained from this study are given below.

1) The IEM provides both necessary and sufficient conditions for stability of the general class of canonical linear periodic systems. The FNIM provides necessary conditions for stability, but is not well suited to providing the sufficient conditions, as further investigation of the governing equations for the FNIM is required under certain circumstances.

2) The IEM is not subject to small parameter restrictions.

3) The IEM is not subject to the numerical problems of the FNIM when dealing with canonical linear periodic systems.

4) The form used for the almost periodic solution, Eq. (14), is amenable to analysis, and it is possible to formulate the infinite matrix $[B]$ before truncation. The truncation level, or equivalently the order of the Fourier series representations of the almost periodic solutions, becomes a user-specified variable in the numerical eigenvalue scheme. The infinite determinant methods proposed by Lindh and Likins[14] and Takahashi[8] require truncating the Fourier series prior to substitution into the governing equations.

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